Introduction to Symbolic Computation for Engineers

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3.-INTRODUCTION TO THE COMPLEXITY OF ALGORITHMS.

- **Let $\mathcal{P}$ be the problem that we want to solve**

- **Let $\mathcal{A}$ be an algorithm used to solve $\mathcal{P}$**

- **$\mathcal{I}_{\mathcal{A}} = \{$ inputs of $\mathcal{A}\}$**

  We define the COMPUTING TIME FUNCTION as:

$$\begin{aligned} t_{\mathcal{A}} : \mathcal{I}_{\mathcal{A}} &\longrightarrow \mathbb{R} \\ x &\longrightarrow t_{\mathcal{A}}(x) \end{aligned}$$

  where $t_{\mathcal{A}}(x)$ represents the number of basic steps needed for executing the algorithm $\mathcal{A}$ to obtain the output $\mathcal{A}(x)$.

- **In general, computing the function $t_{\mathcal{A}}$ is not easy. For this reason we deal with simpler functions.**

Complexity Functions

We consider a partition of the set of inputs of the algorithm:

$$\mathcal{I}_{\mathcal{A}} = \bigcup_{j \in J} \mathcal{I}_{\mathcal{A}}(j)$$

where

- $\mathcal{I}_{\mathcal{A}}(j)$ is a finite set.
- $\mathcal{I}_{\mathcal{A}}(j) \cap \mathcal{I}_{\mathcal{A}}(k) = \emptyset$ if $j \neq k$.

Definition:

- **Maximum Computing Time Function**

$$t^+_{\mathcal{A}} : J \longrightarrow \mathbb{R}$$

$$j \longrightarrow t^+_{\mathcal{A}}(j) = max\{t_{\mathcal{A}}(x) \,|\, x \in \mathcal{I}_{\mathcal{A}}(j)\}$$

- **Minimum Computing Time Function**

$$t^-_{\mathcal{A}} : J \longrightarrow \mathbb{R}$$

$$j \longrightarrow t^-_{\mathcal{A}}(j) = min\{t_{\mathcal{A}}(x) \,|\, x \in \mathcal{I}_{\mathcal{A}}(j)\}$$

- **Average Computing Time Function**

$$t^*_{\mathcal{A}} : J \longrightarrow \mathbb{R}$$

$$j \longrightarrow t^*_{\mathcal{A}}(j) = \sum_{x \in \mathcal{I}_{\mathcal{A}}(j)} p(x) t_{\mathcal{A}}(x)$$

where $p(x)$ is the probability of x.

Remarks:

- **These functions depend on the partition.**
- **In general we analyze $t^+_{\mathcal{A}}$.**

Comparing complexities

Given two algorithms $\mathcal{A}_1$ and $\mathcal{A}_2$ to solve the same problem $\mathcal{P}$ we want to compare their complexity functions $t^+_{\mathcal{A}_1}$ and $t^+_{\mathcal{A}_2}$.

For this purpose we introduce the following notions.

DEFINITION: Let S be a set and $f, g : S \longrightarrow \mathbb{R}$.

- $f \preceq g$ (g **dominates** f) (or f **is of order** g) if

$$\exists\ c \in \mathbb{R}^+ \ /\ \forall\ x \in S\ |f(x)| \leq c|g(x)|.$$

- $f \sim g$ (g and f are **codominant**) $f \preceq g$ and $g \preceq f$.

REMARKS:

- $f \preceq g$ can be denoted as $f = \mathcal{O}(g)$.
- If $0 \leq f \leq g$ then $f \preceq g$.