CHAPTER III: CONICS AND QUADRRICS
I freely confess that I never had a taste for study or research either in physics or geometry except in so far as they could serve as a means of arriving at some sort of knowledge of the proximate causes...for the good and convenience of life, in maintaining health, in the practice of some art...having observed that a good part of the arts is based on geometry, among others the cutting of stone in architecture, that of sundials, and that of perspective in particular.

Gérard Desargues (1591-1661)
1. INTRODUCTION TO THE PROJECTIVE SPACE

1.1 Definitions

Let $V_{n+1}$ be an $(n + 1)$-dimensional vector space. The projective space of dimension $n$ over $V_{n+1}$ is the set of all vector lines of $V_{n+1}$. It is denoted by

$$\mathbb{P}_n(V_{n+1}) = \{ \langle v \rangle \mid v \in V_{n+1} - \{0\} \}.$$

Every vector in $V_{n+1}$ determines a projective point.

Examples

We call the set of vector lines of $\mathbb{R}^3$ real projective plane and we denote it by $\mathbb{P}_2$; this is

$$\mathbb{P}_2 = \mathbb{P}(\mathbb{R}^3) = \{ \langle \overline{v} \rangle \mid \overline{v} \in \mathbb{R}^3 - \{(0, 0, 0)\} \}.$$

We call the set of vector lines of $\mathbb{R}^4$ real projective space and denote it by $\mathbb{P}_3$; this is

$$\mathbb{P}_3 = \mathbb{P}(\mathbb{R}^4) = \{ \langle \overline{v} \rangle \mid \overline{v} \in \mathbb{R}^4 - \{(0, 0, 0, 0)\} \}.$$
1.2 Homogeneous coordinates

Let $\mathbb{P}_n(V_{n+1})$ be a projective space. We say that a family of points $\{<\overline{v}_1>, ..., <\overline{v}_r>\}$ of $\mathbb{P}_n(V_{n+1})$ generate the projective space $\mathbb{P}_n(V_{n+1})$ if the family of vectors $\{\overline{v}_1, ..., \overline{v}_r\}$ generates the vector space $V_{n+1}$.

Let $\mathbb{P}_n(V_{n+1})$ be a projective space. We say that the points $<\overline{v}_1>, ..., <\overline{v}_r>$ of $\mathbb{P}_n(V_{n+1})$ are projectively independent if the vectors $\overline{v}_1, ..., \overline{v}_r$ of $V_{n+1}$ are linearly independent.

Example

Let us consider $\mathbb{P}_2 = \mathbb{P}_2(\mathbb{R}^3)$, then an independent generating family of points of $\mathbb{P}_2 = \mathbb{P}_2(\mathbb{R}^3)$ is formed by three points $X_1 = <\overline{v}_1>, X_2 = <\overline{v}_2>$ and $X_3 = <\overline{v}_3>$ so that the three vectors $\overline{v}_1, \overline{v}_2$ and $\overline{v}_3$ are linearly independent. A point $X = <\overline{w}> \in \mathbb{P}_2$ can be expressed as follows:

$$\overline{w} = \alpha_1 \overline{v}_1 + \alpha_2 \overline{v}_2 + \alpha_3 \overline{v}_3,$$

and the coordinates of $X$ would be $(\alpha_1, \alpha_2, \alpha_3)$. 
If we choose the representative $\lambda \overline{w}$, $\lambda \neq 0$ of $X$, as $X =< \lambda \overline{w} > \in \mathbb{P}_2$ then

$$\lambda \overline{w} = \lambda \alpha_1 \overline{v}_1 + \lambda \alpha_2 \overline{v}_2 + \lambda \alpha_3 \overline{v}_3,$$

and the coordinates of $X$ would be $(\lambda \alpha_1, \lambda \alpha_2, \lambda \alpha_3)$.

We call the class $[\alpha_1, \alpha_2, \alpha_3]$ homogeneous coordinates of the projective point $X$; this is,

$$[\alpha_1, \alpha_2, \alpha_3] = \{ (\lambda \alpha_1, \lambda \alpha_2, \lambda \alpha_3), \text{ with } \lambda \neq 0 \}.$$
1.3 Relationship between affine space and projective space

Let $\mathbb{A}_n$ be an affine space with associated vector space $\mathbb{R}^n$.

Let us consider a coordinate system $\mathcal{R} = \{O, B\}$ of $\mathbb{A}_n$.

Given $X \in \mathbb{A}_n$ with cartesian coordinates $(x_1, \ldots, x_n)$ then

$$(\lambda, \lambda x_1, \ldots, \lambda x_n) \text{ with } \lambda \neq 0$$

is a set of homogeneous coordinates of $X$. We choose $(1, x_1, \ldots, x_n)$ as representative of the homogeneous coordinates of $X$.

**Definition** Given an affine line $P + \langle v \rangle$ were $P \in \mathbb{A}_n$ and $v \in \mathbb{R}^n$ with co-ordinates $(v_1, \ldots, v_n)$ in the basis $B$ then we call $(0, v_1, \ldots, v_n)$ the point at infinity of the affine line.
Definition. Let $\mathbb{A}_n$ be an affine space with associated vector space $\mathbb{R}^n$ with coordinate system $\mathcal{R} = \{O, B\}$. We call the set formed by all the points of $\mathbb{A}_n$ and the points at infinity of $\mathbb{A}_n$ projectivized affine space and denote it by $\overline{\mathbb{A}}_n$; this is

$$
\overline{\mathbb{A}}_n = \mathbb{A}_n \cup \{(0, x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}.
$$

We identify $\overline{\mathbb{A}}_n$ with $\mathbb{P}_n(\mathbb{R}^{n+1})$ in the following way:

$$
\begin{align*}
\overline{\mathbb{A}}_n &\longleftrightarrow \mathbb{P}_n(\mathbb{R}^{n+1}) \\
(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) &\mapsto \langle (x_0, x_1, \ldots, x_n) \rangle, \quad (x_0 \neq 0) \text{ proper points of } \mathbb{P}(\mathbb{R}^{n+1}) \\
(0, x_1, \ldots, x_n) &\mapsto \langle (0, x_1, \ldots, x_n) \rangle, \quad (x_0 = 0) \text{ improper points of } \mathbb{P}(\mathbb{R}^{n+1})
\end{align*}
$$
1.4 Equations of the lines of the projective plane

Let $\mathbb{P}_2$ be the real projective plane.

Given two independent points $P, Q \in \mathbb{P}_2$, we have $P = \langle \overline{v} \rangle$ and $Q = \langle \overline{w} \rangle$ with $\overline{v}, \overline{w} \in \mathbb{R}^3$ linearly independent vectors, the line $r$ that contains $P$ and $Q$ is

$$r = \{ \langle \lambda \overline{v} + \mu \overline{w} \rangle \mid (\lambda, \mu) \neq (0, 0) \}.$$

If the points $P$ and $Q$ have the following homogeneous coordinates:

$$P = [p_0, p_1, p_2], \quad Q = [q_0, q_1, q_2]$$

then a point $X \in r$ if and only if its coordinates $[x_0, x_1, x_2]$ verify the following equations

$$\begin{cases}
\alpha x_0 = \lambda p_0 + \mu q_0 \\
\alpha x_1 = \lambda p_1 + \mu q_1 , \quad (\alpha, \lambda, \mu) \neq (0, 0, 0), \\
\alpha x_2 = \lambda p_2 + \mu q_2
\end{cases}$$
which are called \textit{parametric equations} of the line \( r \) of the projective plane \( \mathbb{P}_2 \).

Equivalently the point \( X = [x_0, x_1, x_2] \in r \) if and only if
\[
a_0x_0 + a_1x_1 + a_2x_2 = 0,
\]
which is the \textit{cartesian equation} of the line that is obtained when we demand the following determinant to be zero:
\[
0 = \begin{vmatrix}
  x_0 & p_0 & q_0 \\
  x_1 & p_1 & q_1 \\
  x_2 & p_2 & q_2
\end{vmatrix}.
\]
1.4.1 Relationship between the lines of the real affine plane and the projective plane.

Let $\mathbb{A}_2$ be the affine plane with coordinate system $\mathcal{R} = \{O, B\}$ and let us consider the line $r$ of the affine plane $\mathbb{A}_2$ with equation $a_0 + a_1x_1 + a_2x_2 = 0$.

Let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ be two points of the line, then two points of the projective plane $[1, p_1, p_2]$, $[1, q_1, q_2]$ determine a line $r$ of the projective plane $\mathbb{P}_2$ with equation $a_0x_0 + a_1x_1 + a_2x_2 = 0$, which is called line of $\mathbb{P}_2$ associated to the affine line $r$.

Reciprocally, given a line $r$ of the projective plane $\mathbb{P}_2$ with equation $a_0x_0 + a_1x_1 + a_2x_2 = 0$. If $p_0 \neq 0$, then the point of the affine plane $\left(\frac{p_1}{p_0}, \frac{p_2}{p_0}\right)$ is in the line $r$ of the affine plane $\mathbb{A}_2$ with equation:

$$a_0 + a_1x_1 + a_2x_2 = 0.$$
Definition. The line that joins two proper points of \( \mathbb{P}_2 \) is called a *proper line* of \( \mathbb{P}_2 \).

Every proper line \( a_0 x_0 + a_1 x_1 + a_2 x_2 = 0 \), determines a point at infinity \([0, -a_2, a_1]\) where \((-a_2, a_1)\) is the direction vector of the line \( r \) of the affine plane \( \mathbb{A}_2 \) with equation \( a_0 + a_1 x_1 + a_2 x_2 = 0 \).

Definition. The line that joins two points at infinity of \( \mathbb{P}_2 \) is called *infinity or improper line* of \( \mathbb{P}_2 \) and has equation \( x_0 = 0 \).
1.5 Equations of projective subspaces of $\mathbb{P}_3$

Let $\mathbb{P}_3$ be the real tridimensional projective space.

1.5.1 Lines in $\mathbb{P}_3$

Let $P, Q$ be two independent points of $\mathbb{P}_3$. Therefore, $P = \langle v \rangle$ and $Q = \langle w \rangle$ with $v, w \in \mathbb{R}^4$ linearly independent vectors. The line $r$ that contains $P$ and $Q$ is

$$r = \{ \langle \lambda v + \mu w \rangle \mid (\lambda, \mu) \neq (0, 0) \}.$$
If the points \( P \) and \( Q \) have the following homogeneous coordinates:

\[
P = [p_0, p_1, p_2, p_3], \quad Q = [q_0, q_1, q_2, q_3]
\]

then a point \( X = [x_0, x_1, x_2, x_3] \in r \) if and only if its coordinates verify the following equations

\[
\begin{align*}
\alpha x_0 &= \lambda p_0 + \mu q_0 \\
\alpha x_1 &= \lambda p_1 + \mu q_1 \\
\alpha x_2 &= \lambda p_2 + \mu q_2 \\
\alpha x_3 &= \lambda p_3 + \mu q_3
\end{align*}
\]

which are called \textit{parametric equations} of the line \( r \) of the projective space \( \mathbb{P}_3 \).
Equivalently the point \( X = [x_0, x_1, x_2, x_3] \) belongs to the line \( r \) of the projective space \( \mathbb{P}_3 \) if and only if

\[
\begin{vmatrix}
  x_0 & p_0 & q_0 \\
  x_1 & p_1 & q_1 \\
  x_2 & p_2 & q_2 \\
  x_3 & p_3 & q_3 \\
\end{vmatrix}
= 2,
\]

from where we obtain the two *cartesian equations* of the line.

**Definition.** The line that joins two proper points of \( \mathbb{P}_3 \) is called a *proper line* of \( \mathbb{P}_3 \). Its equations are the homogeneous equations of an affine line.

**Definition.** The line that joins two improper points of \( \mathbb{P}_3 \) is called *improper or infinity line* of \( \mathbb{P}_3 \).

**Observation.** In \( \mathbb{P}_3 \) there is an infinite number of improper lines.
1.5.2 Planes in $\mathbb{P}_3$

Given three independent points $P = \langle \vec{v} \rangle$, $Q = \langle \vec{w} \rangle$ and $R = \langle \vec{u} \rangle$ of $\mathbb{P}_3$, the plane that contains $P$, $Q$ and $R$ is

$$\pi = \{ \langle \lambda \vec{v} + \mu \vec{w} + \gamma \vec{u} \rangle \mid (\lambda, \mu, \gamma) \neq (0, 0, 0) \}.$$

If the points $P$, $Q$ and $R$ have the following homogeneous coordinates:

$$P = [p_0, p_1, p_2, p_3]$$
$$Q = [q_0, q_1, q_2, q_3]$$
$$R = [r_0, r_1, r_2, r_3]$$

then a point $X = [x_0, x_1, x_2, x_3]$ belongs to the plane $\pi$ of the projective space $\mathbb{P}_3$ if and only if its coordinates verify the following equations

$$\begin{cases}
\alpha x_0 = \lambda p_0 + \mu q_0 + \gamma r_0 \\
\alpha x_1 = \lambda p_1 + \mu q_1 + \gamma r_1 \\
\alpha x_2 = \lambda p_2 + \mu q_2 + \gamma r_2 \\
\alpha x_3 = \lambda p_3 + \mu q_3 + \gamma r_3
\end{cases}, \quad (\alpha, \lambda, \mu, \gamma) \neq (0, 0, 0, 0)$$
which are called *parametric equations* of the plane $\pi$ of the projective space $\mathbb{P}_3$.

Equivalently the point $X = [x_0, x_1, x_2, x_3]$ is contained in the plane $\pi$ of the projective space $\mathbb{P}_3$ if and only if

$$a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0,$$

which is the *cartesian equation* of the plane that is obtained when we force that the following determinant is to be zero:

$$0 = \begin{vmatrix} x_0 & p_0 & q_0 & r_0 \\ x_1 & p_1 & q_1 & r_1 \\ x_2 & p_2 & q_2 & r_2 \\ x_3 & p_3 & q_3 & r_3 \end{vmatrix}.$$ 

**Observations.**

Three proper points determine a *proper plane* of $\mathbb{P}_3$. Its equation is the homogeneous equation of an affine plane.

Three improper points determine an *improper plane* of $\mathbb{P}_3$ which has as
cartesian equation the equation $x_0 = 0$.

Every proper plane determines a line at infinity. Every line at infinity is contained in the infinity plane $x_0 = 0$. 