CHAPTER III: CONICS AND QUADRRICS
2. CONICS

Interseccion of the cone $x^2 + y^2 = z^2$ with planes:

- $z - \frac{x}{3} = 1 \quad \Rightarrow \quad \text{ELLIPSE}$
- $x = 1 \quad \Rightarrow \quad \text{HYPERBOLA}$
- $x - z = 1 \quad \Rightarrow \quad \text{PARABOLA}$
Definition. Given a quadratic form $\omega : \mathbb{R}^3 \rightarrow \mathbb{R}$. The projective conic defined by $\omega$ is the set of points $X \in \mathbb{P}_2(\mathbb{R}^3)$ verifying $\omega(X) = 0$; that is,

$$\bar{C} = \{ X \in \mathbb{P}_2(\mathbb{R}^3) \mid \omega(X) = 0 \}.$$ 

The affine conic defined by $\omega$ is the set of points $X \in \mathbb{A}_2$, $\tilde{X} = (1, x_1, x_2)$, verifying $\omega(\tilde{X}) = 0$; that is,

$$C = \{ X \in \mathbb{A}_2 \mid \omega(\tilde{X}) = 0 \}.$$ 

We have $C \subset \bar{C}$. 
Let $A = (a_{i,j})$ be the matrix of $w$,

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{pmatrix}$$
Let $A = (a_{i,j})$ be the matrix of $w$,

$$
A = \begin{pmatrix}
a_{00} & a_{01} & a_{02} \\
a_{01} & a_{11} & a_{12} \\
a_{02} & a_{12} & a_{22}
\end{pmatrix}
$$

The equation of a conic is given by a second degree polynomial

$$
\bar{C} \equiv \sum_{i=0}^{2} \sum_{j=0}^{2} a_{ij} x_i x_j = 0.
$$

Using matrix notation, the equation of a conic can be written as follows

$$
\bar{C} \equiv X^T A X = 0,
$$

this is

$$
X \in \bar{C} \iff X^T A X = 0.
$$
The equation of the projective conic is:

\[ 0 = \sum_{i=0}^{2} \sum_{j=0}^{2} a_{ij} x_i x_j \]

\[ = a_{00} x_0^2 + a_{11} x_1^2 + a_{22} x_2^2 + 2a_{01} x_0 x_1 \]
\[ + 2a_{02} x_0 x_2 + 2a_{12} x_1 x_2. \]

The equation of the affine conic is obtained substituting \( x_0 = 1 \):

\[ 0 = a_{00} + a_{11} x_1^2 + a_{22} x_2^2 + 2a_{01} x_1 + \]
\[ + 2a_{02} x_2 + 2a_{12} x_1 x_2. \]

We say that a projective conic is *degenerate* if it is reducible (its equation is a product of two polynomials of degree one), otherwise we call it *non-degenerate*. 
Remember.

Definition. A quadratic form $\omega : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a transformation such that there exists a bilinear form $f : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ with $\omega(v) = f(v, v)$, for every $v \in \mathbb{R}^3$.

Result. Given a quadratic form $\omega$ there exists a bilinear form $f$ such that:

1. $f$ is symmetric (this is, $f(u, v) = f(v, u)$),

2. the quadratic form associated to $f$ is $\omega$,

3. $f$ is unique.

We call polar form of $\omega$ the only symmetric bilinear form of $f$ whose quadratic form is $\omega$.

The polar form of a quadratic form is given as follows:

$$f(u, v) = \frac{1}{2}(\omega(u + v) - \omega(u) - \omega(v)).$$

We have:

$$\omega(X) = f(X, X).$$
2.1 Singular points

Definition. Let $\bar{C}$ be a projective conic determined by a quadratic form $\omega$, with polar form $f$ and associated matrix $A$.

- We say that two points $P, Q \in \mathbb{P}_2$ are conjugated if $f(P, Q) = 0$.
- We say that a point $P \in \mathbb{P}_2$ is an autoconjugated point if $\omega(P) = f(P, P) = 0$.
- We say that a point $P \in \mathbb{P}_2$ is a singular point of $\bar{C}$ if it is conjugated with any point of $\mathbb{P}_2$; this is, $f(P, Q) = 0$ for every point $Q \in \mathbb{P}_2$. This is, if
  \[ f(P, Q) = P^T A Q = 0, \quad \forall Q \in \mathbb{P}_2, \]
  or, equivalently,
  \[ P^T A = 0. \]
- We say that a point $P \in \mathbb{P}_2$ is a regular point of $\bar{C}$ if it is not a singular point.
The conic \( \bar{C} \) is **non degenerate, regular or ordinary** if it does not have a singular point.

The conic \( \bar{C} \) is **degenerate or singular** if it has a singular point.

**Examples**

\( \bar{C}_1 \equiv x_0^2 + 2x_1^2 + 3x_1x_2 = 0 \) is a non-degenerate conic, because the homogeneous polynomial of degree 2, \( x_0^2 + 2x_1^2 + 3x_1x_2 = 0 \) is irreducible (we cannot express it as the product of two polynomials of degree 1).

\( C_2 \equiv x_0^2 - 4x_1^2 = 0 \) is degenerate because \( x_0^2 - 4x_1^2 = (x_0 - 2x_1)(x_0 + 2x_1) \); this is, the conic \( C_2 \) decomposes in two lines that intersect.

\( \bar{C}_3 \equiv (x_0 + 2x_1 + 3x_2)^2 = 0 \) is degenerate. The conic \( C_3 \) is a double line.
Observations: Let $\bar{C}$ be a projective conic determined by a quadratic form $\omega$, with polar form $f$ and associated matrix $A$.

1. Let $Sing(\bar{C})$ be the set of singular points of $\bar{C}$, we call it singular locust of $\bar{C}$; this is,

$$Sing(\bar{C}) = \{ X \in \mathbb{P}_2 \mid f(X, Y) = 0, \text{ for every } Y \in \mathbb{P}_2 \}$$
$$= \{ X \in \mathbb{P}_2 \mid AX = 0 \}.$$

We have

$$\dim(Sing(\bar{C})) = 2 - \text{rank}(A).$$

2. If $X \in \mathbb{P}_2$ is a singular point, then $X \in \bar{C}$.

Proof. We have to prove that $\omega(X) = 0$. We have $\omega(X) = f(X, X) = 0$ as $X$ is conjugated with any point, in particular with itself.

3. The line determined by a singular point $X$ and any point that belongs to a conic $Y \in \bar{C}$, is contained in the mentioned conic.
**Proof.** As $X$ is singular, we know that $\omega(X) = 0$ and $f(X, Y) = 0$ and as $Y$ belongs to the conic $\omega(Y) = 0$. Any point of the line determined by $X$ and $Y$ has the form $Z = \lambda X + \mu Y$. We have to check that $\omega(Z) = 0$. We have:

$$
\omega(Z) = \omega(\lambda X + \mu Y) = f(\lambda X + \mu Y, \lambda X + \mu Y)
$$

$$
= f(\lambda X, \lambda X + \mu Y) + f(\mu Y, \lambda X + \mu Y)
$$

$$
= f(\lambda X, \lambda X) + f(\lambda X, \mu Y) + f(\mu Y, \lambda X) + f(\mu Y, \mu Y)
$$

$$
= \lambda^2 f(X, X) + 2\lambda \mu f(X, Y) + \mu^2 f(Y, Y)
$$

$$
= \lambda^2 \omega(X) + 2\lambda \mu f(X, Y) + \mu^2 \omega(Y) = 0.
$$

4. All the points contained in the line joining two singular points are singular.
Proof. Let $Z = \lambda X + \mu Y$ be any point contained in a line formed by two singular points $X$ and $Y$. We have to check $f(Z, T) = 0$, for every $T \in \mathbb{P}_2$. We have:

$$f(Z, T) = f(\lambda X + \mu Y, T) = f(\lambda X, T) + f(\mu Y, T) = \lambda f(X, T) + \mu f(Y, T) = 0.$$

5. If the conic $\tilde{C}$ contains a singular point, then $\tilde{C}$ is formed by lines that contain that point.
2.2 Projective classification of conics

Let \( \bar{C} \) be a conic with associated matrix \( A \).

We will say that the conic \( \bar{C} \) is empty if it has no real points.

<table>
<thead>
<tr>
<th>rankA</th>
<th>sign(A)</th>
<th>Conic</th>
<th>Canonical equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>Empty non-degenerate conic</td>
<td>( x_0^2 + x_1^2 + x_2^2 = 0 )</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>Non empty non-degenerate conic</td>
<td>( x_0^2 + x_1^2 - x_2^2 = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>a singular point</td>
<td>( x_0^2 + x_1^2 = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>pair of lines</td>
<td>( x_0^2 - x_1^2 = 0 )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>double line</td>
<td>((ax_0 + bx_1 + cx_2)^2 = 1)</td>
</tr>
</tbody>
</table>

**Notation:** We name *signature* of \( A \) and we denote it by \( \text{sign}(A) \) to \( |\alpha - \beta| \) where \( \alpha \) is the number of positive eigenvalues of \( A \) and \( \beta \) is the number of negative eigenvalues of \( A \).
2.3 Polarity defined by a conic

Let \( \bar{C} \) be a conic with polar form \( f \) and associated matrix \( A \). Let \( P \in \mathbb{P}_2 \), we call \textit{polar variety} of \( P \) with respect to the conic \( \bar{C} \) to the set of all conjugated points with \( P \); this is,

\[
V_P = \{ X \in \mathbb{P}_2 \mid f(P, X) = 0 \}.
\]

If \( P \) is a singular point, then \( V_P = \mathbb{P}_2 \).

If \( P \) is not a singular point, then \( V_P \) is a line that we denote by \( r_P \) and call \textit{polar line} of \( P \) with respect to the conic \( \bar{C} \).

Therefore, the polar line of a non singular point \( P \in \mathbb{P}_2 \) is the set of points conjugated with \( P \).
2.3.1 Equation of the polar line

If $P$ is a non singular point with coordinates $[p_0, p_1, p_2]$ and the matrix associated to the conic is

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{pmatrix}$$

then

$$r_P = \{ X \in \mathbb{P}_2 \mid P^TAX = 0 \},$$

this is,

$$0 = P^TAX = (p_0, p_1, p_2) \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

$$= (p_0a_{00} + p_1a_{01} + p_2a_{02})x_0 + (p_0a_{01} + p_1a_{11} + p_2a_{12})x_1 + (p_0a_{02} + p_1a_{12} + p_2a_{22})x_2.$$
2.3.2 Pole of a line with respect to a conic \( \tilde{C} \)

**Definition.** Given a line \( r \) of the projective plane \( \mathbb{P}_2 \), we call **pole** of the line \( r \) with respect to the conic \( \tilde{C} \) the point whose polar line is \( r \); this is, \( r_P = r \).

If the equation of the line \( r \) is

\[
   r \equiv u_0x_0 + u_1x_1 + u_2x_2 = U^T X = 0,
\]

with \( U = (u_0, u_1, u_2) \) and \( X = (x_0, x_1, x_2) \),

then \( r_P = r \) if and only if

\[
   P^T A X = U^T X, \text{ for every } X \in \mathbb{P}_2
\]

or equivalently,

\[
   P^T A = U^T \iff A P = U.
\]

If the conic \( \tilde{C} \) is non-degenerate (therefore, \( \det A \neq 0 \)), then \( P = A^{-1} U \).
Theorem. If the polar line of a point $Q$ contains a point $P$, then the polar line of $P$ contains the point $Q$.

This is due to the conjugation condition $f(P, Q) = 0$, which is symmetric in $P$ and $Q$. 
2.3.3 Polarity defined by a conic

As we have seen, given a conic $\bar{C}$ every non singular point $P \in \mathbb{P}_2$ is assigned a line (its polar line) and reciprocally, every line $r$ is assigned a point (its pole).

Definition. We call *polarity defined by a conic* $\bar{C}$ the transformation that makes every point, which is not a singular point of $\bar{C}$, correspond with its polar line, this is,

$$\mathbb{P}_2 \setminus Sing(\bar{C}) \longrightarrow \text{Lines of } \mathbb{P}_2$$

$$P \longmapsto r_P$$

Theorem of polarity defined by a regular conic.

All the polar lines of the points of a line $r$ of $\mathbb{P}_2$, with respect to a regular conic $\bar{C}$, contain the same point which is the pole of $r$. 
2.4 Intersection between a line and a conic

Let \( \bar{C} \) be a projective conic with polar form \( f \) and associated matrix \( A \) and let \( r \) be a projective line that contains the points \( P = [p_0, p_1, p_2] \) and \( Q = [q_0, q_1, q_2] \).

A point \( X \in \mathbb{P}_2 \) is in the intersection between the conic and the line if and only if:

\[
\begin{align*}
X \in r & \iff X = \lambda P + \mu Q \\
X \in \bar{C} & \iff \omega(X) = 0 \\
\omega(\lambda P + \mu Q) & = 0
\end{align*}
\]

The condition \( \omega(\lambda P + \mu Q) = 0 \) is written:

\[
0 = \lambda^2 \omega(P) + 2\lambda \mu f(P, Q) + \mu^2 \omega(Q).
\]

Dividing the above mentioned equation by \( \mu^2 \) and writing \( t = \lambda/\mu \) we obtain the following second degree equation:

\[
0 = \omega(P)t^2 + 2f(P, Q)t + \omega(Q)
\]

with discriminant
\[ \Delta = f(P, Q)^2 - \omega(P)\omega(Q). \]

- If \( f(P, Q) = 0, \omega(P) = 0 \) and \( \omega(Q) = 0 \), then \( P, Q \in \bar{C} \) and, therefore, \( r \subset \bar{C} \). Then the conic is formed by lines.

- If not, every coefficient of the second degree equation \( 0 = \omega(P)t^2 + 2f(P, Q)t + \omega(Q) \) is non zero, then there are two intersection points (the two solutions of the equation).

1. If \( \Delta = f(P, Q)^2 - \omega(P)\omega(Q) > 0 \), the line and the conic intersect in two different proper points. We say that the line is a secant line to the conic.

2. If \( \Delta = f(P, Q)^2 - \omega(P)\omega(Q) = 0 \), the line and the conic intersect in a double point. We say that the line is a tangent line to the conic.

3. If \( \Delta = f(P, Q)^2 - \omega(P)\omega(Q) < 0 \), the line and the conic intersect in two different points at infinity. We say that the line is an exterior line to the conic.
2.4.1 Tangent variety to a conic.

**Definition.** The tangent variety to a conic $\tilde{C}$ at a point $P \in \tilde{C}$, is the set of points $X \in \mathbb{P}_2$ such that the line that joins $P$ and $X$ is tangent to the conic $\tilde{C}$; this is,

$$T_P \tilde{C} = \{ X \in \mathbb{P}_2 \mid \Delta = f(P, X)^2 - \omega(P)\omega(X) = 0 \} = \{ X \in \mathbb{P}_2 \mid f(P, X) = 0 \}.$$

**Remarks**

1. If $P \in \tilde{C}$ is a regular point, then $T_P \tilde{C}$ is a line and, in fact, is the polar line of the point $P$; this is, $T_P \tilde{C} = r_p$.

2. If $P \in \tilde{C}$ is a singular point, then $T_P \tilde{C} = \mathbb{P}_2$. 
3. If \( P \notin \bar{C} \), we can define the *tangent variety* to \( \bar{C} \) at \( P \notin \bar{C} \) as the set of points \( X \in \mathbb{P}_2 \) such that the line that joins \( P \) and \( X \) is tangent to the conic \( \bar{C} \); this is,

\[
T_P\bar{C} = \{ X \in \mathbb{P}_2 \mid \text{line } XP \text{ is tangent to } \bar{C} \}
\]

\[
= \{ X \in \mathbb{P}_2 \mid \Delta = f(P, X)^2 - \omega(P)\omega(X) = 0 \}
\]

\[
= \{ X \in \mathbb{P}_2 \mid f(P, X)^2 = \omega(P)\omega(X) \}.
\]

So \( T_P\bar{C} \) is a degenerate conic that has \( P \) as singular point.