CHAPTER II: AFFINE AND EUCLIDEAN GEOMETRY
3. Euclidean space

A real vector space $E$ is an euclidean vector space if it is provided of an scalar product (or dot product); this is, a bilineal, symmetric and positive-definite map

$$\langle \ , \ \rangle : E \times E \longrightarrow \mathbb{R}.$$ 

We denote a scalar product by $\langle \overline{u}, \overline{v} \rangle$ or $\overline{u} \cdot \overline{v}$ indistinctly.

A scalar product defined in a vector space $E$ allows the definition of a norm as follows:

$$||| : E \longrightarrow \mathbb{R}, \ ||\ v\ ||= \sqrt{\langle v, v \rangle}$$

The angle between two non zero vectors $\overline{u}$ and $\overline{v}$ of an euclidean vector space $E$, is the real number that we will denote by $\widehat{(\overline{u}, \overline{v})}$ such that

$$\cos(\overline{u}, \overline{v}) = \frac{\overline{u}_1 \cdot \overline{u}_2}{||u_1|| \ ||u_2||}.$$
3. AFFINE EUCLIDEAN SPACE

Definition
An affine space \((A, V, \phi)\) is an *euclidean affine space* if the vector space \(V\) is an euclidean vector space.

Notation
We will denote the euclidean vector spaces by \(E\) and the euclidean affine spaces by \((E, E, \phi)\).

Definition
A distance \(d\) inside an affine space \(A\) is a map

\[
d: A \times A \rightarrow \mathbb{R}, (P, Q) \mapsto d(P, Q)
\]

that verifies:

1. \(d\) is positive-definite; this is, \(d(P, Q) \geq 0\) and \(d(P, Q) = 0\) if and only if \(P = Q\).
2. \(d\) is symmetric; this is, \(d(P, Q) = d(Q, P)\).
3. \(d\) verifies the triangle inequality; this is, \(d(P, Q) \leq d(P, R) + d(R, Q)\).
3.1 Orthogonal coordinate systems

An affine coordinate system $\mathcal{R} = \{O; \{\vec{e}_1, \ldots, \vec{e}_n\}\}$ in an euclidean affine space $(\mathbb{E}, E, \phi)$ is called orthogonal (resp. orthonormal), if the basis $B = \{\vec{e}_1, \ldots, \vec{e}_n\}$ of the vector space $V$ is orthogonal (resp. orthonormal).

Change of orthonormal coordinate system

Let $(\mathbb{E}, E, \phi)$ be an euclidean affine space of dimension $n$. Let $\mathcal{R} = \{O; B\}$ and $\mathcal{R}' = \{O'; B'\}$ be two orthonormal coordinate systems of $\mathbb{E}$.

If $O'(a_1, \ldots, a_n)$ and $M(B', B)$ is the matrix of change of basis then the matrix of the change of coordinate system from $\mathcal{R}'$ to $\mathcal{R}$ is:

$$M_f(\mathcal{R}', \mathcal{R}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_1 & \vdots & \ddots & \vdots \\ a_n & & & M(B', B) \end{pmatrix}$$
The following statements hold:

1. The matrix $M(B', B)$ is an orthogonal matrix; this is, $M(B', B)^{-1} = M(B', B)^t$.

2. $\det(M(B', B)) = \pm 1$. If $\det(M(B', B)) = 1$ we say that $B'$ and $B$ have the same orientation and if $\det(M(B', B)) = -1$ we say that $B'$ and $B$ have different orientation.
3.2 Orthogonal affine subspaces

Let \((\mathbb{E}, E, \phi)\) be an euclidean affine space of dimension \(n\). We must remember that, given a vector subspace \(W \subset E\), the set defined as follows:

\[
\{ \overline{v} \in E \mid \overline{v} \cdot \overline{w} = 0 \text{ for every } \overline{w} \in W \}
\]

is a vector subspace of \(E\) that we denote \(W^\perp\) and call \textit{orthogonal subspace to} \(W\). It holds

\[
E = W \oplus W^\perp.
\]

Therefore,

\[
\dim E = \dim W + \dim W^\perp.
\]

Definition

Two affine subspaces \(L_1\) and \(L_2\) of \(\mathbb{E}\) are \textit{orthogonal} if their respective associated vector subspaces \(\overline{L}_1\) and \(\overline{L}_2\) are orthogonal; this is, any vector \(\overline{u} \in \overline{L}_1\) is orthogonal to any vector \(\overline{v} \in \overline{L}_2\). If \(L_1 = P_1 + \langle \overline{u}_1, \ldots, \overline{u}_s \rangle\) and \(L_2 = P_2 + \langle \overline{v}_1, \ldots, \overline{v}_r \rangle\) then \(L_1\) and \(L_2\) are orthogonal if \(\overline{u}_i \cdot \overline{v}_j = 0\) for \(i = 1, \ldots, s\) and \(j = 1, \ldots, r\).
Notice that $L_1 \subseteq L_2$ and therefore,
\[
\dim L_1 + \dim L_2 = \dim L_1 + n - \dim L_2^\perp \leq n.
\]
If $\dim L_1 + \dim L_2 \geq n$, we will say that $L_1, L_2$ are orthogonal if $L_1^\perp$ and $L_2^\perp$ are orthogonal.

**Notation.** If $L_1$ and $L_2$ are orthogonal, we will write $L_1 \perp L_2$. 
Definition
An affine subspace $L'$ with associated vector subspace $\overline{L}'$ is called *orthogonal* to an affine subspace $L$ with associated vector subspace $\overline{L}$ if $\overline{L}$ and $\overline{L}'$ are orthogonal and besides $V = \overline{L} \oplus \overline{L}'$.

Particular cases

1. Two lines $r = P + \langle \bar{v} \rangle$, $r' = P' + \langle \bar{v}' \rangle$ are orthogonal if and only if $\bar{v} \cdot \bar{v}' = 0$.

2. In dimension 3, a line $r = P + \langle \bar{v} \rangle$ is the orthogonal subspace to a plane with associated vector subspace $W$ if $\bar{v}$ is orthogonal to any vector of $W$ (in this case, $V = W \oplus \langle \bar{v} \rangle$).

3. Let $\pi = P + \langle \bar{u}_1, \bar{u}_2 \rangle$ be an affine plane. The line $r = P + \langle \bar{v} \rangle$ is orthogonal to $\pi$ if the vector $\bar{v}$ is orthogonal to vectors $\bar{u}_1$ and $\bar{u}_2$.

4. In dimension 3, a line $r = P + \langle \bar{v} \rangle$ is orthogonal to a plane $\pi = P + \langle \bar{u}_1, \bar{u}_2 \rangle$ if the vector $\bar{v}$ is parallel to the normal vector to the plane; this is, $\bar{v}$ and $\bar{n}$ are parallel, where $\bar{n} = \bar{u}_1 \wedge \bar{u}_2$ and $\wedge$ denotes the cross product in $\mathbb{E}_3$. 
5. In dimension 3, two planes $\pi_1$ and $\pi_2$ are orthogonal if their respective normal vectors are orthogonal.

3.2.1 Orthogonal projection of a point on an affine subspace

Let $L$ be an affine subspace of an euclidean affine space $E$ and let $P$ be a point of $E$ that does not belong to $L$ (this is, $P \in E \setminus L$).

The orthogonal projection of $P$ on $L$ is $P_0$, the point of intersection of the orthogonal subspace to $L$ and containing $P$ with $L$. 

3.3 Distance between two affine subspaces

Let \((E, E, \phi)\) be an euclidean affine subspace of dimension \(n\). Let \(L_1\) and \(L_2\) be two affine subspaces of \(E\). We define the *distance between* \(L_1\) and \(L_2\) as the minimum of the distances between its points; this is,

\[
d(L_1, L_2) = \min \{d(P_1, P_2) \mid P_1 \in L_1 \text{ and } P_2 \in L_2\}.
\]

Notice that if \(L_1 \cap L_2 \neq \emptyset\) then \(d(L_1, L_2) = 0\).

- If \(L_1\) and \(L_2\) are parallel subspaces, let us suppose that \(\overline{L}_1 \subset \overline{L}_2\) then

\[
d(L_1, L_2) = d(P, L_2) = \min \{d(P, P_2) \mid P_2 \in L_2\}
\]

where \(P\) is an arbitrary point of \(L_1\).
If \( L_1 = P_1 + \overline{L}_1 \) and \( L_2 = P_2 + \overline{L}_2 \) are not parallel then we build a subspace \( H \), which is parallel with one of them and contains the other. For example, we can take \( H = P_1 + \overline{L}_1 + \overline{L}_2 \). The subspace \( H \) contains \( L_1 \) and it is parallel with \( L_2 \); therefore,

\[
d(L_1, L_2) = d(H, L_2)
\]

and we are in the first case.

Thus, the problem is just about computing the distance from a point \( P \) to a subspace \( L \).
3.3.1 Distance between a point $P$ and an affine subspace $L$

Let $(\mathbb{E}, E, \phi)$ be an euclidean affine space of dimension $n$. Let $P \in \mathbb{E}$ and let $L = Q + \overline{L}$ be an affine subspace of $\mathbb{E}$, with $P \notin L$. Then, if we call $P_0$ to the orthogonal projection of $P$ on $L$, we have:

$$d(P, L) = d(P, P_0) = ||PP_0||.$$

Now we will study some particular cases of distance between affine subspaces.

Distance between a point $P$ and a hyperplane $H$

Let $P$ be a point with coordinates $(p_1, \ldots, p_n)$ and let $H$ be the hyperplane with cartesian equation $a_1x_1 + \cdots + a_nx_n + b = 0$.

If we denote the orthogonal projection of $P$ on $H$ by $P_0$ we have:

$$d(P, H) = d(P, P_0).$$
Let \( \bar{u} \) be the unit vector normal to the hyperplane; this is,

\[
\bar{u} = \frac{(a_1, \ldots, a_n)}{\sqrt{a_1^2 + \cdots + a_n^2}}
\]

The following formula hold:

\[
d(P, P_0) = |PP_0 \cdot \bar{u}| = \left| (x_1 - p_1, \ldots, x_n - p_n) \cdot \frac{(a_1, \ldots, a_n)}{\sqrt{a_1^2 + \cdots + a_n^2}} \right|
\]

\[
= \frac{|a_1x_1 + \cdots + a_1x_n - (a_1p_1 + \cdots + a_1p_n)|}{\sqrt{a_1^2 + \cdots + a_n^2}}
\]

\[
= \frac{|a_1p_1 + \cdots + a_1p_n + b|}{\sqrt{a_1^2 + \cdots + a_n^2}}
\]
Distance between a point $P$ and a line $r$

Let us consider $P \in \mathbb{E}$ and let $r \equiv Q + \langle \vec{u} \rangle$ be a line in $\mathbb{E}$. By $P_0$ we denote the orthogonal projection of $P$ on $r$, then we have:

$$d(P, r) = d(P, P_0),$$

where $P_0$ is a point of the line $r$ and it holds $\overline{PP_0} \cdot \vec{u} = 0$.

Distance between two skew lines in $\mathbb{E}_3$

Let $r_1 \equiv P_1 + \langle \vec{u}_1 \rangle$ and $r_2 \equiv P_2 + \langle \vec{u}_2 \rangle$ be two lines in $\mathbb{E}_3$. Let us build a plane parallel with one of them, which contains the other one; for example, the plane,

$$\pi \equiv P_2 + \langle \vec{u}_1, \vec{u}_2 \rangle$$

is parallel to the line $r_1$ and contains the line $r_2$. 
Also, let us consider the unit vector normal to the plane $\pi$; this is, the vector 

$$ \overline{u} = \frac{1}{\|\overline{u}_1 \wedge \overline{u}_2\|} \overline{u}_1 \wedge \overline{u}_2 $$

where $\wedge$ denotes the cross product in $\mathbb{E}_3$. We have:

$$ d(r_1, r_2) = d(r_1, \pi) $$

Let us consider the parallelepiped whose edges are vectors $\overline{P_2P_1}$, $\overline{u}_1$ and $\overline{u}_2$.

The volume of the mentioned parallelepiped is the absolute value of the triple product of $\overline{u}_1$, $\overline{u}_2$ and $\overline{P_2P_1}$; this is,

$$ V = |[\overline{u}_1, \overline{u}_2, \overline{P_2P_1}]| = |\overline{P_2P_1} \cdot (\overline{u}_1 \wedge \overline{u}_2)| = \|\overline{P_2P_1}\| \|\overline{u}_1 \wedge \overline{u}_2\| |\cos \alpha| $$

where $\alpha$ is the angle formed by vectors $\overline{P_2P_1}$ and $\overline{u}_1 \wedge \overline{u}_2$. 
The area of the base of the parallelepiped is:

\[ A = \| \bar{u}_1 \wedge \bar{u}_2 \| \]

The distance between \( r_1 \) and \( \pi \) is the height of the above mentioned parallelepiped.

Therefore,

\[ d(r_1, r_2) = d(r_1, \pi) = \frac{\left\| \bar{u}_1, \bar{u}_2, \bar{P}_2 \bar{P}_1 \right\|}{\| \bar{u}_1 \wedge \bar{u}_2 \|} = \left\| \bar{P}_2 \bar{P}_1 \right\| |\cos \alpha| . \]